

EVOLUTION EQUATIONS FOR THE QUARK-MESON TRANSITION

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Abstract:

Evolution equations describe the effect of consecutively integrating out all quantum fluctuations with momenta larger than some infrared cutoff scale k . We develop a formalism for the introduction of collective degrees of freedom at some intermediate scale and derive the corresponding evolution equations. This allows to account for the appearance of bound states at some characteristic length scale as, for example, the mesons in QCD. The vacuum properties, including condensates of composite operators, can be directly inferred from the effective action for $k \rightarrow 0$. We compute in a simple QCD-motivated model with four-quark interactions the chiral condensate $\bar{\psi}\psi$ and the effective action for pions including f_π . A full treatment of QCD along these lines seems feasible but still requires substantial work.

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1 Introduction

A system with many degrees of freedom or a field theory is often described by different relevant excitations at different length scales. A famous example in particle physics is the theory of strong interactions, QCD. At short distances (on scales below 1 fm) the relevant degrees of freedom are quarks and gluons whose interactions are well described by perturbative QCD [1]. The particles which are observed at large scales, however, are mesons and hadrons. The interaction of the pseudoscalar mesons is well modeled by chiral perturbation theory [2]. The corresponding nonlinear σ -model shares the flavour symmetries of perturbative QCD and it is believed that its free phenomenological parameters can ultimately be computed from the QCD Lagrangian. In practice, the transition from one set of degrees of freedom (quarks and gluons) to another (mesons and hadrons) is a difficult task. Its solution would greatly enhance the predictivity of QCD, since the large number of parameters characterizing the meson masses and their low energy interactions would be reduced to the one free mass scale present in QCD plus the values of the short distance quark masses. In this paper we will propose an approach for a description of the “quark-meson transition” which is formulated in continuous spacetime. It is sufficiently general to apply, in principle, to all problems where the relevant degrees of freedom depend on the length scale.

Our approach is based on the concept of the effective average action $\Gamma_k[\varphi]$. $\Gamma_k[\varphi]$ is obtained by integrating out all modes of the quantum field with momenta larger than an infrared cutoff scale, $q^2 > k^2$. For $k \rightarrow 0$ the infrared cutoff is removed and Γ_k becomes the generating functional of the 1PI Green functions. In particular the properties of the vacuum and of excitations around the vacuum can directly be read off from $\Gamma_{k \rightarrow 0}$. For nonvanishing k the integration of quantum fluctuations is only partial. It does not yet include the effects of modes with $q^2 < k^2$. For $k \rightarrow \infty$ (or k equal to some ultraviolet cutoff Λ) the effective average action equals the classical action. Knowledge of the k -dependence of Γ_k therefore allows to interpolate from the classical action to the effective action, including consecutively more and more quantum fluctuations as the scale k is lowered. The scale dependence of the effective average action is described by an exact evolution equation³ ($t = \ln k$)

$$\frac{\partial}{\partial t} \Gamma_k[\varphi] = \frac{1}{2} \text{Tr} \left\{ \left(\Gamma_k^{(2)}[\varphi] + R_k \right)^{-1} \frac{\partial R_k}{\partial t} \right\}. \quad (1.1)$$

³In the present form this equation was first obtained [3] as a renormalization-group improved one-loop equation for the average action. It was later proven by simple manipulation of a functional integral with an infrared cutoff that eq. (1.1) is an exact nonperturbative equation [4]. The equivalence of (1.1) with Polchinski’s version of an exact renormalization group equation [5] was established by means of a Legendre transform [6, 7]. (Note that the “ultraviolet cutoff action” described by [5] does not become the effective action as the cutoff goes to zero, but the generating functional of one-particle reducible Green functions with sources replaced by specific functionals of the fields [8, 9, 10]. A detailed discussion and comparison can be found in [11].) The intentions of Polchinski as well as subsequent authors [6, 8, 12, 13] were mainly aimed towards proofs of perturbative renormalizability. Many earlier versions of exact renormalization group or evolution equations exist in statistical mechanics [14]. Equation (1.1) is surely equivalent to them. Its solutions for $k \rightarrow 0$ are expected to be solutions to the Schwinger-Dyson equation [15].

Here the trace runs typically over momenta and internal indices, and $\Gamma^{(2)}$ denotes the second functional derivative with respect to the fields φ . Within a loopwise expansion, $\Gamma_k = \Gamma_{(0)k} + \Gamma_{(1)k} + \dots$, eq. (1.1) has the iterative solution

$$\Gamma_k = \Gamma_{(0)k} + \frac{1}{2} \ln \text{Det}(\Gamma_{(0)k}^{(2)} + R_k) + \dots \quad (1.2)$$

where the dots denote k -independent counter terms and higher-order terms. Thus equation (1.1) easily reproduces the familiar one-loop result, but the equation is actually exact to all loop orders. The additional infrared cutoff term R_k in the inverse propagator suppresses the propagation of modes with $q^2 < k^2$. The appearance of $\Gamma_k^{(2)}$ on the right-hand side turns equation (1.1) into a complicated partial differential equation for infinitely many variables (typically k and $\varphi(q)$, the Fourier modes of the field φ). Alternatively, equation (1.1) can be viewed as an infinite system of coupled nonlinear differential equations for the flow of the infinitely many couplings needed to parametrize the most general form of Γ_k . In most cases it is impossible to find exact solutions of this equation and its practical use may therefore be questioned. The particularly simple “one-loop” form of (1.1), the simple interpretation of Γ_k both for $k \rightarrow 0$ and $k \rightarrow \infty$ and the representation of Γ_k in terms of a functional integral with constraint [16] allow, however, an educated guess on the relevant variables for a given theory. This permits the “truncation” of Γ_k with an ansatz containing only a finite number of couplings. The existence of useful nonperturbative truncations seems to us the main practical advantage of (1.1) compared to earlier versions of exact renormalization group equations [14, 5], whose approximative solutions in general involved again usual perturbative, large N or ϵ -expansions. In fact, the formal exactness of the evolution equation constitutes by itself not yet any calculational progress compared to more standard perturbative expansion techniques. Only the existence of nonperturbative approximation schemes for its solutions can provide a key for the exploration of nonperturbative physics.

Despite its close resemblance to a one-loop equation the evolution equation (1.1) still contains the full nonperturbative content of a theory. We observe that the r.h.s. of (1.1) is both infrared and ultraviolet finite if an appropriate cutoff R_k is chosen, as for example

$$R_k = \frac{q^2 \exp(-q^2/k^2)}{1 - \exp(-q^2/k^2)} \quad (1.3)$$

$$\lim_{q^2 \rightarrow 0} R_k = k^2 \quad (1.4)$$

No need for an additional specification of an ultraviolet regularization arises in this case. This is replaced by the necessary specification of Γ_k for some sufficiently large scale $k = \Lambda$. (Here Λ may be associated with a physical ultraviolet cutoff in the sense that all quantum fluctuations with $q^2 > \Lambda^2$ are already included in Γ_Λ .) A solution of the exact evolution equation for $k \rightarrow 0$ with initial condition specified by the form of Γ_Λ yields all Green functions of the theory. The possibility to extract useful nonperturbative information even from approximative solutions of eq. (1.1) has already been demonstrated by the successful use of nonperturbative “truncations”: For a $SO(N)$ symmetric scalar field theory the phase structure in

arbitrary dimension d has been correctly reproduced [3], including the Kosterlitz-Thouless phase transition [17] for $d = 2, N = 2$. Critical exponents of the three-dimensional theory have been computed with a few percent accuracy [18]. The high temperature phase transition in the four-dimensional theory was established to be of the second order [19] in contrast to indications of earlier perturbative results. (For more related nonperturbative work on scalar field theories see refs. [20].) Of direct relevance to the present work is the description of bound states [7] through properties of Γ_k : A nonperturbative study of the momentum dependence of the four-point function shows the appearance of a pole in the s -channel from which mass and wave function of the bound state can be extracted. These first encouraging results for the description of nonperturbative physics raise the challenge: Can one tackle with these methods the most important nonperturbative problem in field theory — the behaviour of QCD for low momenta?

One problem for such a project is immediately apparent: For large k ($k \gg 1 \text{ GeV}$) the theory is well described in terms of quarks and gluons and Γ_k should take a relatively simple form in terms of these fields — typically dominated by gauge-invariant kinetic terms and mass terms for the quarks. Due to the growing gauge coupling one expects that the form of Γ_k gets more and more complicated as k comes close to typical scales where strong interaction phenomena manifest themselves, say at k around 1 GeV. At these scales effective four-quark couplings or terms $\sim (F_{\mu\nu}F^{\mu\nu})^2$ do no longer remain small corrections. For k even much smaller than 1 GeV the dominant excitations are mesons and hadrons. A description of the meson and hadron physics in terms of the original quark and gluon degrees of freedom must be extremely complicated: Bound states result in a complicated momentum dependence of the n -point functions and condensates indicate that an expansion of Γ_k around the configuration of vanishing fields presumably becomes meaningless. One clearly needs a transition to express Γ_k in terms of new degrees of freedom more adapted to the physics. In this paper we want to provide the necessary formalism to include in Γ_k collective fields (as appropriate for the mesons and $\bar{\psi}\psi$ condensates). We will again derive an exact evolution equation equivalent to (1.1), but now including the collective fields. Even though on the formal level there is no need to use these new “collective field equations” the necessity of an appropriate nonperturbative truncation scheme makes such an approach almost compulsory. The present work builds on earlier descriptions of collective fields [10] in the framework of Polchinski’s exact renormalization group equations [5].

To be more specific, we expand $\Gamma_k[\varphi]$ in powers of the “fundamental” field φ , with k -dependent one-particle irreducible n point functions $\Gamma_k^{(n)}$ as coefficients. (In the QCD example φ stands for quark and gluon fields.) The n -point functions $\Gamma_k^{(n)}$ will possibly have a complicated dependence on the external momenta. Exact flow equations for each of them can be derived by taking appropriate functional derivatives of (1.1). Let us, for simplicity, consider two-particle bound states or condensates in the following: Informations on possible two-particle bound states are contained in the four-point function $\Gamma^{(4)}$ and are usually extracted by means of the Bethe-Salpeter equation [21]. In our approach we follow the k -dependent

four-point function $\Gamma_k^{(4)}$ from large towards smaller values of k . If the theory under consideration possesses a two-particle bound state in the channel $\varphi_1\varphi_2 \rightarrow \varphi_3\varphi_4$, $\Gamma_k^{(4)}$ will develop a pole in the momentum variable $s = (p_1 + p_2)^2$ for small enough k .⁴ In this case $\Gamma_k^{(4)}$ assumes the form

$$\Gamma_k^{(4)}(p_1, \dots, p_4) \sim -g(p_1, p_2)\tilde{G}(s)g(p_3, p_4) + \Gamma_k'^{(4)}(p_1, \dots, p_4) \quad (1.5)$$

where $g(p_1, p_2)$ corresponds to the (amputated) Bethe-Salpeter wave function and $\tilde{G}(s)$ to the bound state propagator approaching a pole-like dependence on s for small enough k . At the pole $\Gamma_k'^{(4)}(p_i)$ remains a bounded function of the momenta p_1, \dots, p_4 . In ref. [7], where the suitably truncated flow equation for $\Gamma_k^{(4)}$ was integrated numerically for different models, the emergence of the structure described in (1.5) was indeed observed. It was demonstrated that both the wave function g and the propagator $\tilde{G}(s)$ near the pole can be reliably extracted from $\Gamma_k^{(4)}$.

We associate the bound state with a composite operator of the type

$$O(q) = \int \frac{d^4 p_1}{(2\pi)^4} \frac{d^4 p_2}{(2\pi)^4} g(p_1, p_2) \varphi^\dagger(-p_1) \varphi(p_2) (2\pi)^4 \delta^4(q - p_1 - p_2) \quad (1.6)$$

A collective field σ is then introduced for the composite operator O . (In our QCD model φ stands for the quarks and σ denotes the mesons of the linear σ -model.) Since the collective field σ associated with the bound state may also develop a nonvanishing vacuum expectation value, it would be desirable to study the complete effective potential of the field σ and not just its part quadratic in σ . (The latter could be obtained from $\tilde{G}(s)$ at $s = 0$.) This is one of the main motivations for the formalism to be developed in the next sections. This formalism deals with the effective average action $\Gamma_k[\varphi, \sigma]$ involving one (or more) collective fields σ . It requires, at a certain scale k_φ , to replace $\Gamma_{k_\varphi}[\varphi]$ by (cf. eq. (5.11))

$$\Gamma_{k_\varphi}[\varphi, \sigma] = \Gamma_{k_\varphi}[\varphi] + \frac{1}{2} O^\dagger[\varphi] \tilde{G} O[\varphi] - \sigma^\dagger O[\varphi] + \frac{1}{2} \sigma^\dagger \tilde{G}^{-1} \sigma \quad (1.7)$$

The complete effective action for both fundamental and collective fields $\Gamma[\varphi, \sigma]$ is then obtained after evolving $\Gamma_k[\varphi, \sigma]$ from $k = k_\varphi$ down to $k = 0$ with the help of a modified flow equation discussed in sects. 3, 4.

If the four-point function $\Gamma_k^{(4)}$ has, at a scale $k \sim k_\varphi$, developed a pole-like structure as in (1.5), a corresponding term will appear in the part quartic in φ in $\Gamma_{k_\varphi}[\varphi]$ on the r.h.s. of (1.7). The operator $O[\varphi]$ can then be chosen appropriately such that the pole-like structure in $\Gamma_{k_\varphi}^{(4)}$ is cancelled by the second term on the r.h.s. of (1.7). With this choice the operator $O[\varphi]$ contains the two-particle wave function of $g(p_1, p_2)$ of eq. (1.5). After the cancellation of the pole-like structure in $\Gamma_k^{(4)}$ the evolution of $\Gamma_k[\varphi, \sigma]$ down to $k = 0$ can proceed with new "initial condition" $\Gamma_{k_\varphi}^{(4)} \simeq \Gamma_{k_\varphi}'^{(4)}$. The remaining part of the four point function stays then bounded

⁴Even though we work in Euclidean space where the evolution equation involves $\Gamma^{(n)}$ with all $p_i^2 \geq 0$ and $s \geq 0$, the final expressions for the Green functions can be analytically continued to Minkowski space and $s < 0$.

(unless further bound states appear). In this way it becomes apparent how the introduction of the collective field σ can remove a dominant strong interaction with a complicated momentum dependence from the Green functions of the fundamental fields. The price to pay is an effective average action with more fields and therefore more possible couplings. Even though at the scale k_φ the effective average action starts only with terms linear and quadratic in the collective field σ as in eq.(1.7), this property will not be preserved for smaller values of k . The flow equation typically generates interaction terms for σ as well. Despite the apparent complication through the presence of additional fields, the approximate (truncated) flow equations may become much simpler. Since we have now explicitly included the relevant degrees of freedom, the important physics may be described in terms of fewer couplings than needed for a reliable description of Γ_k in terms of the fundamental fields alone!

This paper is organized as follows: In sect. 2 we develop the formalism of the effective average action including collective fields, and we derive exact flow equations. In sect. 3 we present the flow equations in a version where “fundamental” and collective fields are treated on equal footing. Various strategies how to add or remove effective degrees of freedom in dependence on the characteristic length scale of the problem are discussed in sect. 4. For the case of QCD this gives the formalism how to replace quark degrees of freedom by meson degrees of freedom. In sects. 5 and 6 we finally apply our formalism to a QCD motivated model with four-quark interactions. We integrate out the quarks and obtain an effective potential for the linear σ -model. This allows a determination of f_π and the chiral condensate $\bar{\psi}\psi$. Conclusions and prospects for quantitative improvements of our results are contained in sect. 7.

2 Exact evolution equations with composite operators and collective fields

Our starting point is the scale-dependent generating functional for the connected Green functions ⁵

$$W_{k,0}[J, K] = \ln \int \mathcal{D}\chi \exp -S_k[\chi] \quad (2.1)$$

$$S_k[\chi] = S[\chi] + \Delta_k S[\chi] - J^\dagger \chi - K^\dagger \tilde{G} O[\chi] - \frac{1}{2} K^\dagger \tilde{G} K \quad (2.2)$$

Here χ^α are the (bosonic) quantum fields of the theory⁶ with action $S[\chi]$ and J_α^* the associated sources with

$$J^\dagger \chi = J_\alpha^* \chi^\alpha \quad (2.3)$$

The indices α include momenta q^μ , possible Lorentz indices for vector and tensor fields as well as internal indices. (For example, the action for free scalar fields in d Euclidean dimensions reads $S = \frac{1}{2} \chi^\dagger q^2 \chi = \frac{1}{2} \int \frac{d^d q}{(2\pi)^d} \chi_a^*(q) (q_\mu q^\mu) \chi^a(q)$. In this

⁵The second index 0 of $W_{k,0}$ refers to an optional infrared cutoff for collective fields to be discussed in sect. 3.

⁶The generalization for fermionic fields is straightforward [22] and will be given in Appendix C.

notation functional derivatives obey $\delta\chi^a(q)/\delta\chi^b(q') = \delta_b^a\delta(q, q')$ where $\delta(q, q') = (2\pi)^d\delta^d(q - q')$ such that $\text{Tr}_q\delta(q, q') = 1$.)

Let us neglect first the last two terms in S_k which involve K . Then the only modification of the standard definition of the generating function $W[J]$ is the addition of an infrared cutoff which is quadratic in χ :

$$\Delta_k S[\chi] = \frac{1}{2}\chi^\dagger R_k \chi \quad (2.4)$$

The matrix $(R_k)_\beta^\alpha$ will be chosen such that it suppresses the contribution of fluctuations with momenta $q^2 \ll k^2$ in the functional integral (2.1). We require that the infrared cutoff $\Delta_k S$ preserves all symmetries of the theory. A typical choice⁷ for R_k for the case of scalar fields with internal indices a, b is

$$(R_k)^a_b(q, q') = R_k(q)\delta_b^a\delta(q, q') \quad (2.5)$$

$$R_k(q^2) = \frac{q^2 \exp(-q^2/k^2)}{1 - \exp(-q^2/k^2)} \quad (2.6)$$

For $q^2 \ll k^2$ one finds $R_k \approx k^2$ and the infrared cutoff scale k acts like an additional mass. On the other side R_k is exponentially small for $q^2 \gg k^2$ and does therefore not affect the functional integration of modes with $q^2 \gg k^2$. If desired, however, $R_k(q^2)$ could also be chosen such that it incorporates an UV cutoff Λ . Then only fluctuations with $q^2 < \Lambda^2$ are included in the functional integral (see sect. 5). A very intuitive picture arises if we combine the classical kinetic term from $S[\chi]$ with $\Delta_k S$ which leads to a lowest order propagator $G_k^{(0)}(q)$ of the form⁸

$$G_k^{(0)}(q) = (q^2 + R_k)^{-1} = \frac{1 - \exp(-q^2/k^2)}{q^2} \quad (2.7)$$

We see that the propagation of modes with $q^2 \gg k^2$ remains essentially unchanged whereas modes with $q^2 \ll k^2$ cease to propagate.

We observe that for $k \rightarrow 0$ and $K = 0$ the functional $W_{k,0}$ approaches the standard definition of $W[J]$

$$\begin{aligned} \lim_{k \rightarrow 0} \Delta_k S &= 0 \\ \lim_{k \rightarrow 0} W_{k,0}[J, 0] &= W[J] \end{aligned} \quad (2.8)$$

Performing a Legendre transformation of $W_{k,0}$ for $K = 0$ and subtracting the infrared cutoff term yields the effective average action $\Gamma_{k,0}[\varphi]$ which becomes the generating functional for the 1PI Green functions for $k \rightarrow 0$ [4]

$$\Gamma_{k,0}[\varphi] = -W_{k,0}[J, 0] + J^\dagger \varphi - \frac{1}{2}\varphi^\dagger R_k \varphi$$

⁷For precision calculations it is convenient to incorporate in $R_k(q)$ appropriate wave function renormalization constants and possibly also mass terms.

⁸If $R_k(q^2)$ involves an UV cutoff Λ , it can be chosen such that $G_{k,\Lambda}^{(0)}$ has the form $G_{k,\Lambda}^{(0)} = ((\exp -q^2/\Lambda^2) - \exp(-q^2/k^2))/q^2$.

$$\begin{aligned}
&= \tilde{\Gamma}_{k,0}[\varphi] - \frac{1}{2}\varphi^\dagger R_k \varphi \\
\varphi^\alpha &= \frac{\delta W_{k,0}[J, 0]}{\delta J_\alpha^*}
\end{aligned} \tag{2.9}$$

The dependence of $\Gamma_{k,0}$ on the scale k is described by the exact evolution equation (1.1) mentioned in the introduction.

We now want to consider composite operators $O^i[\chi]$ for which we have introduced appropriate sources K_i^* in (2.1), with

$$K^\dagger \tilde{G} O[\chi] = K_i^* \tilde{G}^i_j O^j[\chi] \tag{2.10}$$

Here \tilde{G} typically plays the role of a propagator with an appropriate dimensional factor such that the sources have a standard normalization. In momentum space \tilde{G} is a function of q^2 . For simplicity of notation it is often convenient to use rescaled operators

$$\tilde{O}[\chi] = \tilde{G} O[\chi] \tag{2.11}$$

such that the coupling to the sources takes the standard form $K^\dagger \tilde{O}$. We also have used the freedom [23, 21] to add in (2.2) a field-independent term quadratic in K with a matrix \tilde{G}^i_j . The role of this term will become more apparent later. The generating functional for the connected Green functions for the fields χ as well as for the composite operator obtains again in the limit $k \rightarrow 0$

$$\lim_{k \rightarrow 0} W_{k,0}[J, K] = W[J, K] \tag{2.12}$$

For example, the expectation value of the operator \tilde{O}^i reads

$$\langle \tilde{O}^i \rangle = \frac{\delta W}{\delta K_i^*} \Big|_{J=0, K=0} \tag{2.13}$$

and the connected two-point function is⁹

$$\langle \tilde{O}^i \tilde{O}_j^* \rangle - \langle \tilde{O}^i \rangle \langle \tilde{O}_j^* \rangle = \frac{\delta^2 W}{\delta K_i^* \delta K_j} \Big|_{J=0, K=0} - \tilde{G}^i_j \tag{2.14}$$

Knowledge of the functional $W[J, K]$ therefore allows the computation of various “condensates” and correlations between “condensates”.

More generally, we will define for an arbitrary scale k and arbitrary source J, K classical fields φ^α and σ^i ,

$$\begin{aligned}
\frac{\delta W_{k,0}}{\delta J_\alpha^*} &= \varphi^\alpha \\
\frac{\delta W_{k,0}}{\delta K_i^*} &= \sigma^i,
\end{aligned} \tag{2.15}$$

⁹We have chosen a normalization appropriate for real composite operators where \tilde{O}_i^* and \tilde{O}_i are not independent. Our conventions also cover complex fields and operators if index summations over internal indices include negative values [4].

where the composite or collective field σ^i represents the composite operator \tilde{O}_i . We combine the fundamental and composite fields and their associated sources in single vectors

$$\begin{aligned} j_m^* &= (J_\alpha^*, K_i^*) \\ \psi^m &= (\varphi^\alpha, \sigma^i) \\ \frac{\delta W_{k,0}}{\delta j_m^*} &= \psi^m \end{aligned} \quad (2.16)$$

The generalized two-point function reads then

$$G_n^m = \frac{\delta^2 W_{k,0}}{\delta j_m^* \delta j^n} \quad (2.17)$$

We now perform a Legendre transform with respect to both “fundamental” and “composite” sources J and K

$$\tilde{\Gamma}_{k,0}[\psi] + W_{k,0}[j] - j^\dagger \psi = 0 \quad (2.18)$$

$$\frac{\delta \tilde{\Gamma}_{k,0}}{\delta \psi^m} = j_m^* \quad (2.19)$$

$$G_n^m \frac{\delta^2 \tilde{\Gamma}_k}{\delta \psi_n^* \delta \psi^p} = \delta_p^m \quad (2.20)$$

In particular, the values of φ and σ extremizing $\tilde{\Gamma}_{0,0}$ determine the expectation values of χ and $\tilde{O}[\chi]$ and the second functional derivative of $\tilde{\Gamma}_{0,0}$ at the minimum gives the exact inverse propagator.

The exact evolution equation or flow equation describing the dependence of $\tilde{\Gamma}_{k,0}$ on the infrared cutoff scale k is now easily obtained. We notice that the only dependence of $W_{k,0}$ on k arises through $\Delta_k S$ which is quadratic in χ . The derivative of $W_{k,0}$ with respect to $t = \ln k$ can therefore be expressed in terms of the Green function G

$$\begin{aligned} \frac{\partial}{\partial t} \tilde{\Gamma}_{k,0|_\psi} &= -\frac{\partial}{\partial t} W_{k,0|_j} = \frac{\partial}{\partial t} \langle \Delta_k S \rangle \\ &= \frac{1}{2} \frac{\partial}{\partial t} (R_k)^\alpha{}_\beta \langle \chi_\alpha^* \chi^\beta \rangle \\ &= \frac{1}{2} \left(\frac{\partial R_k}{\partial t} \right)^\alpha{}_\beta (G^\beta{}_\alpha + \langle \chi^\beta \rangle \langle \chi_\alpha^* \rangle) \end{aligned} \quad (2.21)$$

Subtracting from $\tilde{\Gamma}_{k,0}$ the infrared cutoff

$$\Gamma_k = \tilde{\Gamma}_k - \frac{1}{2} \varphi^\dagger R_k \varphi \quad (2.22)$$

and using the identity (2.20) gives the final form of the exact evolution equation for the effective average action Γ_k including composite fields

$$\frac{\partial}{\partial t} \Gamma_{k,0} = \frac{1}{2} \text{Tr} \left\{ \frac{\partial R_k}{\partial t} (\Gamma_{k,0}^{(2)} + R_k)^{-1} \right\} \quad (2.23)$$

It expresses the scale dependence of $\Gamma_{k,0}$ in terms of its second functional derivative

$$\left(\Gamma_{k,0}^{(2)}\right)_n^m = \frac{\delta^2 \Gamma_{k,0}}{\delta \psi_m^* \delta \psi^n} \quad (2.24)$$

We have extended in (2.23) the matrix R_k to act on vectors ψ , with $(R_k)^i_j = 0$, $(R_k)^i_\alpha = 0$, $(R_k)^\alpha_i = 0$. Only the (α, β) components of R_k (the ones corresponding to the fundamental fields φ^α) depend on k in this formulation. Even though only the α, β components of $(\Gamma_{k,0}^{(2)} + R_k)^{-1}$ contribute therefore in (2.23), the (i, j) and (i, α) components of $\Gamma_{k,0}^{(2)}$ are relevant since they enter in forming the inverse of $\Gamma_{k,0}^{(2)} + R_k$.

For $k \rightarrow \infty$ $\Gamma_{k,0}[\varphi, \sigma]$ approaches

$$\Gamma_{k \rightarrow \infty, 0}[\varphi, \sigma] = S[\varphi] + \frac{1}{2} O^\dagger[\varphi] \tilde{G} O[\varphi] - \sigma^\dagger O[\varphi] + \frac{1}{2} \sigma^\dagger \tilde{G}^{-1} \sigma. \quad (2.25)$$

In a theory with an ultraviolet cutoff Λ (cf. footnote 7), the average action becomes equal to the r.h.s. of eq. (2.25) for $k = \Lambda$. Solving the evolution equation (2.23) with this initial condition interpolates from the short distance physics described by Γ_Λ to the physics at longer distances described by Γ_k , $k < \Lambda$. In this process the quantum fluctuations with momenta $\Lambda^2 > q^2 > k^2$ are integrated out. Γ_k for $k \rightarrow 0$ therefore amounts to the complete effective action of the quantum field theory.

In summary, we have presented here an extended form of the exact evolution equation (1.1). It uses additional composite fields σ . A solution of $\Gamma_k[\varphi, \sigma]$ for $k \rightarrow 0$ encodes all information of the original procedure, namely all 1PI Green functions for the fundamental fields χ . In addition, it contains the complete information on expectation values of composite operators $\tilde{O}[\chi]$ and their correlations. At first sight the use of an enlarged matrix $\Gamma_{k,0}^{(2)}$ may seem to be an additional complication which is the price for gaining additional information about Green functions for composite operators. It may happen, however, that physically important pieces in Γ_k have a rather complicated form when expressed in terms of the “fundamental fields” φ , but become simple once they are written in terms of composite operators. In this case the use of more fields (φ, σ) may result in a considerable simplification of calculations. This applies in particular to models with condensates of collective fields and/or bound states. For the theory of strong interactions we may associate φ with the gluon and quark degrees of freedom and σ with operators like $\bar{\psi}\psi$, $F_{\mu\nu}F^{\mu\nu}$ or mesons, hadrons, and glueballs. The low energy part of the effective action is expected to become much simpler when expressed in terms of σ rather than of φ .

3 The two-field formalism

In a variety of physical situations there is no clear distinction between fundamental and composite fields. This applies, for example, if one deals with processes with typical momentum transfers much smaller than the mass scale characteristic for the structure of the bound state. The structure cannot be resolved in this case, and the bound state appears as an ordinary particle. An example are versions of the

standard model where the Higgs scalar is a bound state. In situations of this type one would like to treat fundamental fields and composite fields on a completely equal footing (at least at low energies) such that standard field-theoretical methods as perturbation theory can be applied. For this purpose it is instructive to write the functional $W_{k,0}[J, K]$ in terms of a functional integration over both fundamental and collective degrees of freedom. We use the identity (choosing \tilde{G} such that $\tilde{G}^\dagger = \tilde{G}$)

$$1 = N \int \mathcal{D}\rho \exp \left\{ -\frac{1}{2}(\rho^\dagger - K^\dagger \tilde{G} - O^\dagger[\chi] \tilde{G}) \tilde{G}^{-1} (\rho - \tilde{G} K - \tilde{G} O[\chi]) \right\} \quad (3.1)$$

with N an irrelevant constant to be omitted in the following. Insertion in (2.1) yields

$$W_{k,0}[J, K] = \ln \int D\chi D\rho \exp -S_k[\chi, \rho] \quad (3.2)$$

with

$$\begin{aligned} S_k[\chi, \rho] &= S[\chi] - J^\dagger \chi + \Delta_k S[\chi] \\ &+ \frac{1}{2} O^\dagger[\chi] \tilde{G} O[\chi] - \rho^\dagger O[\chi] - K^\dagger \rho + \frac{1}{2} \rho^\dagger \tilde{G}^{-1} \rho. \end{aligned} \quad (3.3)$$

We see that $W_{k,0}[J, K]$ has now the same form of a theory with additional fields and interactions. In particular, S_k is linear in the source K which motivates the term $\sim K^\dagger \tilde{G} K$ in the original formulation (2.2). Compared to the formulation in terms of only fundamental fields χ we have introduced new fields ρ with inverse propagator \tilde{G}^{-1} and interactions with the fundamental field $\sim \rho^\dagger O[\chi]$. For the example of an operator O quadratic in χ one obtains a cubic interaction as depicted in fig. 1. One also has an additional term $\sim O^\dagger \tilde{G} O$ involving only the fields χ . It has the form of an exchange of ρ in the tree approximation (fig. 2) and can be used to cancel terms in $S[\chi]$ with a pole structure as discussed in the introduction and in more detail in sections 4 and 5.

Inside the one-loop diagram corresponding to the r.h.s. of the flow equation (2.23) also collective fields appear. Their presence is induced by the inversion of $(\Gamma_{k,0}^{(2)} + R_k)$ in the enlarged space with indices (i, α) , (cf. the discussion following eq. (2.24)). The effective propagator \tilde{G} of the collective fields ρ does not yet include an infrared regulator; thus it is possible that the r.h.s. of the flow equation (2.23) contains, in a given truncation, an infrared divergence. This situation can especially arise in the case of dynamical symmetry breaking, where some collective degrees of freedom become massless Goldstone bosons.¹⁰ In the formulation (3.3) it becomes easy to introduce an additional infrared cutoff (with scale \tilde{k}) for the collective field ρ . We can generalize $W_{k,0}$ to $W_{k,\tilde{k}}$ by adding to S_k (3.3) a term

$$\tilde{\Delta}_{\tilde{k}} S[\rho] = \frac{1}{2} \rho^\dagger \tilde{R}_{\tilde{k}} \rho. \quad (3.4)$$

¹⁰In practice one might sometimes consider a truncation of $\Gamma_{k,0}$ where vertices generating “internal” collective fields are discarded, see sect. 5. Here, however, we consider the general case.

The matrix $(\tilde{R}_{\tilde{k}})^i_j$ should have similar properties as $(R_k)^\alpha_\beta$. In particular it should vanish for $\tilde{k} \rightarrow 0$, and all eigenvalues should diverge for $\tilde{k} \rightarrow \infty$ (or $\tilde{k} \rightarrow \Lambda$). Correspondingly, $\tilde{\Gamma}_{k,\tilde{k}}$ is defined by the analogue of (2.18) and $\Gamma_{k,\tilde{k}}$ reads

$$\Gamma_{k,\tilde{k}} = \tilde{\Gamma}_{k,\tilde{k}} - \frac{1}{2}\varphi^\dagger R_k \varphi - \frac{1}{2}\sigma^\dagger \tilde{R}_{\tilde{k}} \sigma. \quad (3.5)$$

We note

$$\frac{\delta W_{k,\tilde{k}}}{\delta K_i^*} = \langle \rho^i \rangle = \sigma^i \quad (3.6)$$

with the expectation value evaluated now with the action $S_k[\chi, \rho] + \tilde{\Delta}_{\tilde{k}} S[\rho]$. The evolution equation for the k -dependence of $\Gamma_{k,\tilde{k}}$ at fixed \tilde{k} has the same form as (2.23) with $(R_k)^i_j$ replaced by $(\tilde{R}_{\tilde{k}})^i_j$, which is now different from zero but independent of k . (Only the denominator on the r.h.s. of (2.23) changes due to the new definition (3.5). As in the previous section, $(R_k)^\alpha_\beta$ and $(\tilde{R}_{\tilde{k}})^i_j$ are combined to an enlarged matrix $R_{k,\tilde{k}}$.) The dependence of $\Gamma_{k,\tilde{k}}$ on $\tilde{t} = \ln \tilde{k}$ can also be derived in complete analogy to (2.23)

$$\frac{\partial}{\partial \tilde{t}} \Gamma_{k,\tilde{k}} = \frac{1}{2} \text{Tr} \left\{ \frac{\partial R_{k,\tilde{k}}}{\partial \tilde{t}} (\Gamma_{k,\tilde{k}}^{(2)} + R_{k,\tilde{k}})^{-1} \right\} \quad (3.7)$$

Here $R_{k,\tilde{k}}$ acts on vectors (φ, σ) as defined above and only $\frac{\partial}{\partial \tilde{t}} (R_{k,\tilde{k}})^i_j = \frac{\partial}{\partial \tilde{t}} (\tilde{R}_{\tilde{k}})^i_j$ is different from zero in this case. In particular, we may identify k and \tilde{k} and define

$$\Gamma_k[\varphi, \sigma] = \Gamma_{k,k}[\varphi, \sigma] \quad (3.8)$$

The evolution equation for the k -dependence of Γ_k is again given by (2.23) since

$$\frac{\partial}{\partial t} \Gamma_k = \frac{\partial}{\partial t} \Gamma_{k,\tilde{k}|\tilde{k}=k} + \frac{\partial}{\partial \tilde{t}} \Gamma_{k,\tilde{k}|\tilde{k}=k} \quad (3.9)$$

Now both $(R_k)^\alpha_\beta$ and $(R_k)^i_j$ depend on k . The evolution equation for the two sorts of fields φ and σ has exactly the same form as the original equation (1.1).

In the limit $k \rightarrow \Lambda, \tilde{k} \rightarrow \Lambda$ (Λ may be infinity) the functional integrals defining $W_k = W_{k,k}$ and Γ_k are easily solved. The term quadratic in the fields $\sim R_k$ diverges, the classical approximation becomes exact and one obtains the same result as in (2.25):

$$\Gamma_\Lambda[\varphi, \sigma] = S[\varphi] + \frac{1}{2} O^\dagger[\varphi] \tilde{G} O[\varphi] - \sigma^\dagger O[\varphi] + \frac{1}{2} \sigma^\dagger \tilde{G}^{-1} \sigma \quad (3.10)$$

The equivalence of the initial condition (3.10) with the original formulation in terms of fundamental fields only is readily established by solving the field equations for σ and inserting in Γ_Λ

$$\begin{aligned} \frac{\delta \Gamma_\Lambda}{\delta \sigma}|_{\sigma_0} &= 0 \\ \sigma_0 &= \tilde{G} O[\varphi] = \tilde{O}[\varphi] \\ \Gamma_\Lambda[\varphi, \sigma_0] &= S[\varphi] \end{aligned} \quad (3.11)$$

In the opposite limit $k \rightarrow 0, \tilde{k} \rightarrow 0$ the infrared cutoff term $\sim R_k$ vanishes and Γ_0 becomes $\Gamma_{0,0}$ of sect. 2, the generating functional for the 1PI Green functions for the fields φ and σ . Inserting the solution of the field equation for σ which minimizes $\Gamma_0[\varphi, \sigma]$

$$\frac{\delta \Gamma_0}{\delta \sigma}|_{\sigma_0} = 0 \quad (3.12)$$

we recover the generating functional for the 1PI Green functions for the fundamental field χ (since $K = 0$)

$$\Gamma[\varphi] = \Gamma_0[\varphi, \sigma_0[\varphi]] \quad (3.13)$$

The full generating functional $\Gamma_0[\varphi, \sigma]$ contains all information on the vacuum expectation of the operator $\tilde{O}[\chi]$ as well as all n -point functions of this operator. The procedure for extracting these quantities is straightforward and given explicitly in appendix A.

4 Scale-dependent degrees of freedom

The relevant degrees of freedom often depend on the length scale. For the example of QCD one would like to describe the short-distance physics in terms of gluons and quarks and the long-distance physics in terms of mesons and hadrons. In sect. 3 we have introduced the effective average action $\Gamma_{k,\tilde{k}}$ depending on two scales k and \tilde{k} , which will allow us to describe a smooth transition between different sorts of relevant degrees of freedom. Consider first the case that below some momentum scale k_σ we want to describe the physics uniquely in terms of the collective fields σ . For example, one may want to describe QCD below $k_\sigma \approx 500$ MeV by a linear or nonlinear σ -model for the pseudoscalar mesons. In our approach this can be done very naturally by first fixing the infrared regulator for the collective fields $\tilde{k} = k_\sigma$ and solving the evolution equation for the infrared regulator for the fundamental fields $k \rightarrow 0$. The resulting effective action for composite fields $\Gamma_{\tilde{k}}[\sigma]$ is given by

$$\Gamma_{\tilde{k}}[\sigma] = \Gamma_{0,\tilde{k}}[\varphi_0[\sigma], \sigma] \quad (4.1)$$

where

$$\frac{\delta \Gamma_{0,\tilde{k}}}{\delta \varphi}[\varphi_0[\sigma], \sigma] = 0 \quad (4.2)$$

for $\tilde{k} = \tilde{k}_\sigma$. It can be used as a starting point for integrating out the “mesons” with momenta $q^2 < k_\sigma^2$. The low momentum fluctuations of the mesons are not yet accounted for in $\Gamma_{\tilde{k}}[\sigma]$ due to the existence of an effective infrared cutoff (3.4). They will be included by evolving $\Gamma_{\tilde{k}}[\sigma]$ to $\Gamma_0[\sigma]$ using the evolution equation (3.7). We observe that R is now nonvanishing only for the composite part ($R^\alpha_\beta = 0$ since $k = 0$, $R^i_j = (\tilde{R}_{\tilde{k}})^i_j$) and that $\Gamma_{0,\tilde{k}}^{(2)}$ becomes block-diagonal in composite and fundamental fields

$$\frac{\delta^2 \Gamma_{0,\tilde{k}}}{\delta \sigma \delta \varphi}|_{\varphi_0, \sigma} = 0 \quad (4.3)$$

provided $\varphi_0[\sigma]$ (5.2) is independent of σ . (If the fundamental fields φ assume no vacuum expectation values, the solutions of eq. (4.2) will simply be $\varphi_0[\sigma] = 0$.) Then the evolution equation (3.7) only involves $\Gamma_{\tilde{k}}^{(2)}[\sigma]$ on the r.h.s. and has exactly the same form as (1.1), now expressed in terms of composite fields. This statement generalizes to an arbitrary dependence of φ_0 on σ as briefly explained in appendix B. We conclude that the solution of $\Gamma_{k,\tilde{k}}$ for $k \rightarrow 0$ at fixed \tilde{k} can be used for a complete replacement of the fundamental variables φ by the composite variables σ . This procedure amounts to integrating out the fluctuations φ in a background σ .¹¹ The same procedure can actually also be applied to the problem how to integrate out only one sort of fundamental fields (without reference to collective degrees of freedom). One may introduce a variable infrared cutoff with scale k only for the fields to be integrated out, while the remaining fields have a cutoff with scale Λ . Lowering k to zero at fixed Λ then only includes quantum fluctuations of the fields to be integrated out, and (4.1), (4.2) with $\tilde{k} \equiv \Lambda$ give the effective action for the remaining fields σ .

Under certain circumstances it may be convenient to consider both fundamental and composite degrees of freedom on an equal footing. This is done most easily by a study of the evolution equation (3.9) where $\tilde{k} = k$. An example are top condensate models at scales much below the composite scale [24], which have already been treated by methods closely related to the present one [25]. Treating the scalar bound state (the composite Higgs doublet) similar to the fundamental fermions will reproduce most easily the results of the perturbative standard model since for $k = \tilde{k}$ the fermion and scalar loops described by the formal expression (3.9) are treated in the same way. This is not guaranteed in the (“asymmetric”) version (2.23) where $\tilde{k} = 0$. Since the standard contribution of the scalar loops is missing, a relatively complicated truncation of $\Gamma_{k,0}$ may be needed in order to reproduce even the standard one-loop result for the running of weak couplings. Another typical application of the running of $\Gamma_{k,k}$ is the transition region between the quark description and meson description in QCD at scales between 500 MeV and 2 GeV.

Bound states are, in general, characterized by a typical “composite” scale. At length scales much shorter than the inverse composite scale, they play no particular role. For large enough k not only the introduction of a second scale \tilde{k} seems cumbersome, but the whole formalism with composite degrees of freedom seems sometimes not to be very well adapted. One does not want to describe asymptotic short-distance QCD by dealing explicitly with mesons! This problem can be coped with by “switching on” the collective degrees of freedom only for the evolution at k smaller than some scale k_φ . This can be achieved by an appropriate choice of the composite operator $O[\chi]$. Actually, already the presence of a wave function $g(p_1, p_2)$ in $O[\varphi]$ (see eq.(1.6)), which decays for large momenta p_i^2 , will suppress the contributions of the new terms to the flow of $\Gamma_k[\varphi, \sigma]$ for large k^2 . This effect is much enhanced by the introduction of exponentially decaying functions of p_i^2 into the operator $O[\varphi]$. Let us represent O as a functional of the Fourier modes $\chi(q)$ in

¹¹The actual calculations of the present paper in sects. 5,6 will not involve internal propagators of the σ -field, thus the introduction of a scale k_σ is not necessary.

the form

$$O[\chi] = \hat{O}[f_{k_\varphi}(q)\chi(q)] \quad (4.4)$$

$$f_{k_\varphi}(q) = \exp(-q^2/2k_\varphi^2) \quad (4.5)$$

Typically \hat{O} may be some polynomial of $f_{k_\varphi}(q)\chi(q)$ involving appropriate wave functions — see the next section. More general forms of \hat{O} are also allowed, provided it does not contain an exponentially growing momentum dependence. The appearance of the exponential suppression factor f_{k_φ} in the coupling between ρ and χ (3.3) will then effectively “switch off” all effects of the composite fields for $q^2 \gg k_\varphi^2$. Indeed, for $k > Ak_\varphi$ with A sufficiently large (say $A = 10$) all contributions from composite fields in the evolution equation (2.23) are suppressed. This allows us to compute $\Gamma_{Ak_\varphi}[\varphi]$ by integrating the evolution equations for $Ak_\varphi < k < \Lambda$ without using the formalism with composite fields (solving (1.1)). On the other hand, the modified operator $O[\varphi]$ still describes appropriately the “bound state” for momenta $p_i^2 < k_\varphi^2$ of the fundamental fields. At $k = Ak_\varphi$ one may switch to the two-field formalism. The initial terms on the r.h.s. of eqs. (2.25) or (3.10), which formally have to be introduced for $k \rightarrow \infty$ (or $k = \Lambda$), did not affect the evolution of Γ_k for $k > Ak_\varphi$, and can thus be introduced at $k = Ak_\varphi$ only:

$$\Gamma_{Ak_\varphi, \tilde{k}}[\varphi, \sigma] = \Gamma_{Ak_\varphi}[\varphi] + \frac{1}{2}O^\dagger[\varphi]\tilde{G}O[\varphi] - \sigma^\dagger O[\varphi] + \frac{1}{2}\sigma^\dagger \tilde{G}^{-1}\sigma \quad (4.6)$$

At this stage one has several options for the infrared cutoff \tilde{k} for the collective fields σ , as described before: One can put it equal to zero right away, or one can put it equal to Ak_φ and treat it as a variable independent of k subsequently (using eq. (2.23)), or one can identify it with k both at the starting point $k = Ak_\varphi$ and concerning the subsequent evolution, using eq. (3.9). Note that the r.h.s. of eq. (4.6) does not depend on the choice of \tilde{k} ; the dependence of $\Gamma_{Ak_\varphi, \tilde{k}}$ on \tilde{k} manifests itself only in the form of the evolution equations. The collective degrees of freedom will now affect the evolution of $\Gamma_{k, \tilde{k}}[\varphi, \sigma]$ for $k < Ak_\varphi$ due to the disappearance of the exponential suppression factor (4.5) in the $\sigma - \varphi$ coupling.

The concrete choice of the operators $O[\varphi]$ and the form of the propagator \tilde{G} for the collective fields σ depends on the problem under consideration. For the case of propagating bound states it has been sketched in the introduction. Let us concentrate in the following on a “two-particle bound state” showing up in the four-point function $\Gamma_k^{(4)}$. Our starting point is the effective action $\Gamma_k[\varphi]$ without collective fields σ . After the integration of the corresponding flow equations (1.1) down to a certain “bound state” scale it is assumed that a pole-like structure as in (1.5) has emerged as part of the four-point function $\Gamma_k^{(4)}$. It is now desirable to choose the operator $O[\varphi]$ and the propagator \tilde{G} such that this pole-like structure within $\Gamma_k^{(4)}$ gets cancelled; then the collective field σ associated with the operator $O[\varphi]$ corresponds to the bound state degree of freedom generated by the dynamics of the theory. (Formally no error is made if $O[\varphi]$ is not chosen appropriately, but then informations on the bound state are still contained partially in the remaining part of the four-point function of the fundamental fields.) An obvious possible choice is

to identify the ansatz for $\tilde{G}(q^2)$ in eqs. (2.2), (3.1) and (4.6) with the dynamically generated $\tilde{G}(s)$ in eq. (1.5) and to take the Fourier components of $O[\varphi]$ as

$$O(q) = \int \frac{d^4 p_1}{(2\pi)^4} \frac{d^4 p_2}{(2\pi)^4} f_{k_\varphi}(p_1) f_{k_\varphi}(p_2) g(p_1, p_2) \varphi(p_1) \varphi(p_2) (2\pi)^4 \delta^4(p_1 + p_2 - q) \quad (4.7)$$

Here the scale k_φ should be somewhat above the scale where a pole-like structure in $\Gamma_k^{(4)}$ emerges. This ensures that f_{k_φ} does not switch off the collective fields in the momentum range where they are needed. On the other hand the presence of f_{k_φ} justifies the introduction of the additional terms in eq. (4.6) only at $k = Ak_\varphi$ and not already at $k \rightarrow \infty$ (or $k = \Lambda$) as in eqs. (2.25), (3.10). In practice it might also be useful to try modified forms of the operator $O[\varphi]$ and \tilde{G} introduced at $k = Ak_\varphi$; the optimal choice is the one which completely prevents the appearance of a pole-like structure in $\Gamma_k^{(4)}$ for $k \ll k_\varphi$.

The methods discussed in this section allow for describing smoothly the transition from perturbative QCD to the nonlinear σ -model. Variations of the transition scale Ak_φ and the different options for the infrared cutoff \tilde{k} for the collective fields would allow for a variety of checks. Obviously, much work needs to be done before such a program can be implemented. We nevertheless will demonstrate the viability of our method by the investigation of a simplified model in sect. 5. We finally mention that the restriction of the formalism developed in the last sections to scalar fields is by no means necessary. The generalization for the inclusion of (chiral) fermions is presented in appendix C. Inclusion of gauge fields requires more thought but seems not to encounter insurmountable difficulties [9, 13, 26].

5 A QCD-motivated model

So far we have developed the general formalism for the description of bound states and collective fields in the context of the average action in a very general but necessarily also rather abstract way. In this section we want to give an example which shows how these ideas work in practice. Our aim is a demonstration of feasibility of our program rather than an attempt to compute with precision. We nevertheless take a model which is inspired by QCD. It contains sufficiently many details such that it can account for mesons and chiral condensates and can later be extended to full QCD (see the discussion in sect. 7). We work with massless quarks even though the introduction of quark mass terms poses no particular technical problem.

We start with a theory with only fermion fields ψ . They correspond to massless quarks and play the role of the fundamental fields φ of the last sections. We have in mind a version of QCD, where the gluons have already been integrated out. As a result we expect that, among others, a (generally nonlocal) four-quark interaction $\Gamma_\Lambda^{(4)}(p_i)$ has been generated at a scale $\Lambda \sim 1.5$ GeV. Concerning the form of $\Gamma_\Lambda^{(4)}(p_i)$ we assume that for $p_i^2 < \Lambda^2$ it depends just on the Mandelstam variable t such that it corresponds to the sum of a linear and a confining potential in the nonrelativistic limit. We consider an expansion of the effective action up to fourth order in the quark fields, and we neglect the running of the effective quark propagator. (Due

to chiral symmetry the quarks will remain massless anyhow; we only neglect the running of the quark wave function normalization.) Thus we just have to integrate the evolution equation for $\Gamma_k^{(4)}$ with the above $\Gamma_\Lambda^{(4)}$ as boundary condition at $k = \Lambda$. Now we observe indeed that $\Gamma_k^{(4)}$ at some small scale $\sim k_\varphi$ develops a pole-like structure as in (1.5). It allows us to read off the wave function $g(p_1, p_2)$ and the collective field propagator $\tilde{G}(s)$.

Next we neglect the difference between the bound-state scale, k_φ and Ak_φ , and construct $\Gamma[\psi, \bar{\psi}, \sigma]$ at the scale k_φ according to the rule eq. (4.6). (Here we put $f_{k_\varphi} = 1$ for simplicity.) By this procedure the model is transformed into a linear σ -model with momentum dependent Yukawa couplings between mesons and quarks. The part of Γ_k involving the scalars starts at this scale only with a quadratic term and Yukawa couplings as given by the r.h.s. of eq. (4.6). At scales k below k_φ we focus our attention on the effective potential for the collective field σ . It will be generated after integrating out the quark fields $\psi, \bar{\psi}$ with momenta q with $0 \leq q^2 \leq k_\varphi^2$, due to the (momentum dependent) Yukawa coupling. The dominant effect is the “quadratic running” [16, 22] of the scalar mass term. The fermion fluctuations induce a negative mass term at the origin ($\sigma = 0$) and trigger therefore the spontaneous breaking of chiral symmetry. We neglect contributions to the effective potential due to internal σ -lines. This allows to compute the effective potential from a simple “one-loop formula” (eq. (6.19) below) in terms of the parameters present in $\Gamma_{k_\varphi}[\psi, \bar{\psi}, \sigma]$.

To be more precise, we describe the fundamental quark fields by four component Dirac spinors $\psi_i^a, \bar{\psi}_a^i$ where the flavour index a runs from $1 \dots N_f$, $i = 1 \dots N_c$ is the colour index and we omit the spinor indices. The effective action contains terms quadratic and quartic in the quark fields, and is taken to be invariant under $SU(N_c)$ and chiral $U(N_f)_V \otimes U(N_f)_A$ symmetries:

$$\begin{aligned} \delta_V \psi^a &= i\theta_{Vb}^a \psi^b, & \delta_A \psi^a &= i\gamma_5 \theta_{Ab}^a \psi^b \\ \delta_V \bar{\psi}_a &= -i\bar{\psi}_b \theta_{Va}^b, & \delta_A \bar{\psi}_a &= i\bar{\psi}_b \theta_{Aa}^b \gamma_5. \end{aligned} \quad (5.1)$$

The quadratic part describes the inverse quark propagator, and a possible k dependent wave function normalization is put equal to 1 (we contract over the not explicitly written spinor indices):

$$\Gamma_{2,k} = \int \frac{d^4 q}{(2\pi)^4} \bar{\psi}_a^i(q) \not{q} \psi_i^a(q) \quad (5.2)$$

Even in the presence of the vector and axial vector symmetries (5.1) a number of different four-quark interactions can be written down. We restrict ourselves to one particular spin, flavour and colour structure:

$$\begin{aligned} \Gamma_{4,k} &= \frac{1}{2} \int \prod_{l=1}^4 \left(\frac{d^4 p_l}{(2\pi)^4} \right) (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4) \lambda_k(p_1, p_2, p_3, p_4) \\ &\times \{ [\bar{\psi}_a^i(-p_1) \psi_i^b(p_2)] [\bar{\psi}_b^j(p_4) \psi_j^a(-p_3)] - [\bar{\psi}_a^i(-p_1) \gamma_5 \psi_i^b(p_2)] [\bar{\psi}_b^j(p_4) \gamma_5 \psi_j^a(-p_3)] \}. \end{aligned} \quad (5.3)$$

This structure colour singlets with nontrivial flavour in the s -channel (they will later correspond to the mesons) and flavour singlets transforming as colour singlets and

octets in the t -channel (as mediated, for example, by a one-gluon exchange). As usual the kinematic variables s and t are defined here by

$$\begin{aligned} s &= (p_1 + p_2)^2 = (p_3 + p_4)^2 \\ t &= (p_1 - p_3)^2 = (p_2 - p_4)^2. \end{aligned} \quad (5.4)$$

The contractions of spinor indices are indicated by brackets, and the invariance under (5.1) is readily checked.

The sum of $\Gamma_{2,k}$ and $\Gamma_{4,k}$ has now to be inserted into the fermionic version of the flow equation:

$$\partial_t \Gamma_k = -\text{Tr} \left\{ \frac{\partial R_{kF}}{\partial t} (\Gamma_k^{(2)} + R_{kF})^{-1} \right\}. \quad (5.5)$$

We use here¹²

$$R_{kF}(q) = \not{q} \frac{1 - \exp(-q^2/\Lambda^2) + \exp(-q^2/k^2)}{\exp(-q^2/\Lambda^2) - \exp(-q^2/k^2)} \quad (5.6)$$

such that the sum of the terms $\Gamma_k^{(2)} + R_{kF}$ gives

$$\frac{\not{q}}{\exp(-q^2/\Lambda^2) - \exp(-q^2/k^2)} + O(\bar{\psi}\psi). \quad (5.7)$$

We are interested in the terms $\sim (\bar{\psi}\psi)^2$ of eq. (5.5), which contribute to the running of the coefficient $\lambda_k(p_1, p_2, p_3, p_4)$ of $\Gamma_{4,k}$. After a corresponding expansion of the r.h.s. of eq. (5.5) the equation describing the running of λ_k assumes the schematic form shown in fig. 3. Here the lower inner line denotes the regularized fermionic propagator as derived from the inverse of $\Gamma_k^{(2)} + R_k$,

$$G_{2,k}(q) = \frac{\exp(-q^2/\Lambda^2) - \exp(-q^2/k^2)}{\not{q}}, \quad (5.8)$$

and the upper crossed line its derivative with respect to $t = \ln k$. (In fig. 3 we have not shown a similar diagram with the cross on the lower line.) Actually, inserting the ansatz (5.3) for $\Gamma_{4,k}$ into the r.h.s. of eq. (5.5), also different spin, flavour, and colour structures than the one of eq. (5.3) are generated on the l.h.s. of eq. (5.5). They are put to zero in our truncation. Since only the retained contribution is proportional to a combinatorial colour factor N_c our truncated evolution equation becomes exact in the leading order in a $1/N_c$ expansion. After taking the various combinatorial factors into account and performing the trace over spinor indices, the flow equation for λ_k becomes

$$\begin{aligned} \partial_t \lambda_k(p_1, p_2, p_3, p_4) = & \\ & -\frac{8N_c}{k^2} \int \frac{d^4 q}{(2\pi)^4} \lambda_k(p_1, p_2, q, -q + p_1 + p_2) \lambda_k(q, -q + p_1 + p_2, p_3, p_4) \\ & \left[\frac{q^\mu (q - p_1 - p_2)_\mu}{q^2} \exp(-(q - p_1 - p_2)^2/k^2) (\exp(-q^2/\Lambda^2) - \exp(-q^2/k^2)) \right. \\ & \left. + (q \rightarrow p_1 + p_2 - q) \right] \end{aligned} \quad (5.9)$$

¹²For this choice $R_k(q)$ does not uniformly approach zero for $k \rightarrow 0$. The divergence for $R_k(q \rightarrow 0)$ is of no relevance in the present context

In the model under consideration this flow equation for λ_k will be integrated with a boundary condition for λ_k at $k = \Lambda \simeq 1.5$ GeV, which is assumed to be generated after the gluons in QCD have been integrated out. Its form is motivated by the sum of one gluon exchange and a linearly rising potential, and it is assumed to depend on the variable $t = (p_1 - p_3)^2$ only. After Fierz-transforming the one-gluon exchange diagram in spin and colour space and extracting the contribution proportional to the spin, colour, and flavour structure (5.3), this boundary condition reads

$$\lambda_\Lambda(p_1, p_2, p_3, p_4) = \frac{2\pi\alpha_s}{t} + \frac{8\pi\lambda}{t^2} + D(t) \quad (5.10)$$

Here α_s is the strong gauge-coupling constant which we take $\alpha_s \sim .3$, for the string tension we use $\lambda \sim .18$ GeV², and $D(t)$ is a distribution with support at $t = 0$ only, which is related to a constant in the potential [27] and will be given implicitly later.

Solving the evolution equation (5.9) numerically - for details see the next section - we see indeed a pole-like structure as in eq. (1.5) appearing in λ_k . At the scale k_φ , where the collective fields are introduced, two conditions should be fulfilled: first, λ_{k_φ} should approximately factorize in the form

$$\lambda_{k_\varphi}(p_1, p_2, p_3, p_4) = g(p_1, p_2)\tilde{G}(s)g(p_3, p_4) \quad (5.11)$$

Second, λ_{k_φ} and hence Γ_{4,k_φ} should not yet have become extremely large: Γ_4 appears on the right hand sides of the evolution equations for the higher N point functions Γ_N with $N > 4$. If Γ_4 becomes large, these higher N point functions necessarily become large as well, and their neglect becomes a very questionable approximation in this regime. (Generally, the introduction of the here neglected suppression factor f_{k_φ} of eq. (4.5) and the use of eq. (4.6) at some scale $Ak_\varphi > k_\varphi$ avoid a possible conflict between these two conditions.) Within the present model we find, however, that both conditions can be met simultaneously to a reasonable extend at a scale $k = k_\varphi \sim .63$ GeV (see sect. (6) and fig. (4)).

As a result of eq. (5.11) Γ_{4,k_φ} can be written as

$$\begin{aligned} \Gamma_{4,k_\varphi} = & \frac{1}{2} \int \prod_l \left(\frac{d^4 p_l}{(2\pi)^4} \right) (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4) \\ & \left\{ [\bar{\psi}_a^i(-p_1)\psi_i^b(p_2)]g(p_1, p_2)\tilde{G}(s)g(-p_4, -p_3)[\bar{\psi}_b^j(p_4)\psi_j^a(-p_3)] \right. \\ & \left. - [\bar{\psi}_a^i(-p_1)\gamma_5\psi_i^b(p_2)]g(p_1, p_2)\tilde{G}(s)g(-p_4, -p_3)[\bar{\psi}_b^j(p_4)\gamma_5\psi_j^a(-p_3)] \right\} \\ & + \Gamma'_{4,k_\varphi}. \end{aligned} \quad (5.12)$$

Now we perform the step to switch to the effective average action for quarks and mesons $\Gamma_{k_\varphi,0}[\psi, \bar{\psi}, \sigma]$. Since we only consider the quark contributions to the evolution of the scalar part of the effective average action, there is no advantage to introduce a nonvanishing infrared cutoff \tilde{k} for the collective fields σ in the present approximation. We choose composite operators which read in momentum space

$$\begin{aligned} O_a^b[\psi, \bar{\psi}; q] &= -i \int \frac{d^4 p}{(2\pi)^4} g(p, q-p) \bar{\psi}_a^i(-p) \psi_i^b(q-p) \\ O_a^{(5)b}[\psi, \bar{\psi}; q] &= - \int \frac{d^4 p}{(2\pi)^4} g(p, q-p) \bar{\psi}_a^i(-p) \gamma_5 \psi_i^b(q-p) \end{aligned} \quad (5.13)$$

where the real function $g(p_1, p_2)$ is normalized

$$g(0, 0) = 1 \quad (5.14)$$

such that $\tilde{G}(s)$ has dimensions $\sim 1/s$. Inserting eq. (5.13) in (4.6), one obtains

$$\begin{aligned} \Gamma_{k_\varphi, 0}[\psi, \bar{\psi}, \sigma] &= \Gamma_{k_\varphi}[\psi, \bar{\psi}] \\ &+ \frac{1}{2} \int \frac{d^4 q}{(2\pi)^4} \left\{ O_a^b(q) \tilde{G}(q^2) O_b^a(-q) + O_a^{(5)b}(q) \tilde{G}(q^2) O_b^{(5)a}(-q) \right. \\ &\quad + \tilde{\sigma}_b^a(q) \tilde{G}^{-1}(q^2) \tilde{\sigma}_a^b(-q) + \tilde{\sigma}_b^{(5)a}(q) \tilde{G}^{-1}(q^2) \tilde{\sigma}_b^{(5)a}(-q) \\ &\quad \left. - 2\tilde{\sigma}_b^a(q) O_a^b(-q) - 2\tilde{\sigma}_b^{(5)a}(q) O_a^{(5)b}(-q) \right\}. \end{aligned} \quad (5.15)$$

It is convenient to define σ by

$$\begin{aligned} \tilde{\sigma} &= \frac{1}{2}(\sigma + \sigma^\dagger), \\ \tilde{\sigma}^{(5)} &= \frac{i}{2}(\sigma - \sigma^\dagger) \end{aligned} \quad (5.16)$$

such that the field σ_b^a transforms under chiral flavour transformations $U_L(N) \times U_R(N)$ as an (\bar{N}, N) representation

$$\delta \sigma_b^a = i(\theta_V^a{}_c - \theta_A^a{}_c) \sigma_b^c - i\sigma_b^c (\theta_V^c{}_b + \theta_A^c{}_b) \quad (5.17)$$

After using $\Gamma_{k_\varphi}[\psi, \bar{\psi}] = \Gamma_{2, k_\varphi}[\psi, \bar{\psi}] + \Gamma_{4, k_\varphi}[\psi, \bar{\psi}]$ and eq. (5.12) for $\Gamma_{4, k_\varphi}[\psi, \bar{\psi}]$, the terms quartic in ψ in $\Gamma_{k_\varphi}[\psi, \bar{\psi}, \sigma]$ cancel and we are left with

$$\begin{aligned} \Gamma_{k_\varphi}[\psi, \bar{\psi}, \sigma] &= \int \frac{d^4 q}{(2\pi)^4} \left\{ \bar{\psi}_a^i(q) \not{q} \psi_i^a(q) + \frac{1}{2}(\sigma^\dagger)_b^a(q) \tilde{G}^{-1}(q^2) \sigma_b^a(q) \right. \\ &\quad \left. + i \int \frac{d^4 p}{(2\pi)^4} g(-q, q-p) \bar{\psi}_a^i(q) \left[\frac{1}{2}(1 + \gamma_5) \sigma_b^a(p) + \frac{1}{2}(1 - \gamma_5) (\sigma^\dagger)_b^a(-p) \right] \psi_i^b(q-p) \right\} \end{aligned} \quad (5.18)$$

We have neglected here the term Γ'_{4k_φ} in eq. (5.12) which will be dropped in the following.

Except for the momentum dependence of g the effective average action for composite fields Γ_{k_φ} describes a standard fermion-scalar theory with Yukawa coupling but without scalar self-interactions.

In order to compute $\Gamma_{0,0}[0, \sigma]$, we use the flow equation for $\Gamma_{k,0}[0, \sigma]$ in the form of the fermionic version (C.5) of eq. (2.23)

$$\partial_t \Gamma_{k,0}[0, \sigma] = -\text{Tr} \left\{ \frac{\partial R_{kF}}{\partial t} (\Gamma_{k,0}^{(2)}|_{\psi=0} + R_{kF})^{-1} \right\} \quad (5.19)$$

We observe that $\Gamma_{k,0}^{(2)}|_{\psi=0}$ does not mix the fermionic and bosonic parts and we therefore only need to consider the second functional derivative with respect to the fermions. Apart from the inverse free quark propagator the only contribution to $\Gamma_{k,0}^{(2)}|_{\psi=0}$ originates from the Yukawa coupling $\sim \sigma \bar{\psi} \psi$, which we assume to be

independent of k in the form specified by (5.18). In this approximation the relevant part of $\Gamma_{k,0|\psi=0}^{(2)}$ becomes independent of k and eq. (5.19) is easily integrated. The result is

$$\Gamma_{k,0}[0, \sigma] = -\text{Tr} \ln \left(\frac{\Gamma^{(2)} + R_k}{\Gamma^{(2)} + R_{k_\varphi}} \right) + \frac{1}{2} \sigma^\dagger \tilde{G}^{-1} \sigma, \quad (5.20)$$

where we implemented the boundary condition from (5.18)

$$\Gamma_{k_\varphi,0}(0, \sigma) = \frac{1}{2} \sigma^\dagger \tilde{G}^{-1} \sigma \quad (5.21)$$

in agreement with eq. (4.6).

The effective potential for the field σ or its effective wave function normalization can now easily be obtained from eq. (5.20) for $k = 0$: In the case of the effective potential $\Gamma^{(2)}$ on the r.h.s. of eq. (5.20) has to be evaluated for constant configurations of the field σ , and the momentum dependent collective field propagator $\tilde{G}^{-1}(q^2)$ is only needed at $q^2 = 0$. In the case of the effective wave function normalization Z_k contributions arise both from an expansion of the $\text{Tr} \ln$ term up to second order in the external momentum ($O(q^2)$), and from a corresponding expansion of $\tilde{G}^{-1}(q^2)$. Note that this second contribution is already present at the scale k_φ according to eq. (5.21); thus an often used "compositeness condition" $Z_k = 0$ [24, 25] does not hold at $k = k_\varphi$.

As we will see in the next section, this effective potential exhibits indeed a non-trivial minimum for a real diagonal σ . This expectation value is invariant under the vectorlike flavour symmetry, but carries nonvanishing axial charges. Our model therefore describes dynamical chiral symmetry breaking in this QCD-motivated fermionic theory.

6 Numerical computations and results

In this section we describe in more detail the procedure outlined in the last section, and we give an account of the results of the numerical solution of the evolution equation. The technique for integrating numerically the flow equation (5.9) follows the one developed in [7]. The function λ_k can only depend on six independent Lorentz-invariant products of the momenta, which we denote by

$$\begin{aligned} s &= (p_2 + p_1)^2 = (p_3 + p_4)^2, & t &= (p_1 - p_3)^2 = (p_2 - p_4)^2 \\ v_1 &= p_1^2, & v_2 &= p_2^2, & v_3 &= p_3^2, & v_4 &= p_4^2. \end{aligned} \quad (6.1)$$

Then we switch from the function $\lambda_k(p_i)$ to its Laplace transform with respect to the Lorentz five invariants t, v_1, v_2, v_3, v_4 :

$$\lambda_k(p_i) = \int_0^\infty dl_t dl_1 dl_2 dl_3 dl_4 C_k(s, l_i) e^{-l_t t - l_1 v_1 - l_2 v_2 - l_3 v_3 - l_4 v_4} \quad (6.2)$$

After inserting (6.2) into (5.9) and using

$$\frac{e^{-\frac{q^2}{\Lambda^2}} - e^{-\frac{q^2}{k^2}}}{q^2} = \int_{1/\Lambda^2}^{1/k^2} d\alpha e^{-\alpha q^2} \quad (6.3)$$

the flow equation for $C_k(s, l_i)$ can easily be derived, since the d^4q integration in (5.9) becomes Gaussian. The dependence of $C_k(s, l_i)$ on the variables l_i is then discretized, thus $C_k(s, l_i)$ becomes a function living on a five-dimensional lattice. The integration of the flow equation corresponds to an algorithm for updating the lattice for each step from k^2 to $k^2 + \Delta k^2$. The boundary condition for $C_\Lambda(s, l_i)$, with $\Lambda = 1.5$ GeV, corresponding to (5.10) reads

$$C_\Lambda(s, l_i) = \tilde{C}(l_t) \delta(l_1) \delta(l_2) \delta(l_3) \delta(l_4) \quad (6.4)$$

with

$$\tilde{C}(l_t) = 8\pi\lambda l_t + 2\pi\alpha_s \quad (6.5)$$

As discussed in [7], a vanishing of $\tilde{C}(l_t)$ for $l_t > l_{t_{max}}$ corresponds to a vanishing of the potential $V(r)$ in ordinary space for $r \gg (l_{t_{max}})^{1/2}$, and is related to the distribution $D(t)$ in eq. (5.10). With the help of the formulas in [7] a direct relation between $l_{t_{max}}$ and the depth d of the linear part λr of the potential can be derived:

$$l_{t_{max}} = \frac{\pi d^2}{4\lambda^2} \quad (6.6)$$

This finite value for $l_{t_{max}}$ specifies implicitly the distribution $D(t)$ in (5.10). We will use (6.6) for $l_{t_{max}}$ with $d = 1$ GeV.

With these methods and parameters, we compute $C_k(s, l_i)$ on a lattice with 50 points in the l_t direction, and 10 points in the l_1, l_2, l_3 and l_4 directions. For $k = k_\varphi \sim .63$ GeV we observe that $C_{k_\varphi}(s, l_i)$ approaches a factorized form

$$C_{k_\varphi}(s, l_i) \simeq \delta(l_t) \tilde{g}(l_1, l_2) \tilde{G}(s) \tilde{g}(l_3, l_4) \quad (6.7)$$

and $\tilde{G}(s)$ becomes large for $s \rightarrow 0$. In fig. 4 we show, as a function of the scale k , $\tilde{G}(s = 0)$ as a full line. If $C(s, l_i)$ can be written as a product as on the r.h.s. of (6.7), a measure f defined by $f = [C(s, l_1 = 1, l_2 = 1, l_j) \cdot C(s, l_1 = 0, l_2 = 0, l_j)] / [C(s, l_1 = 1, l_2 = 0, l_j) \cdot C(s, l_1 = 0, l_2 = 1, l_j)]$ should satisfy $f \simeq 1$. In fig. 4 we also show this measure f as a dotted line. Thus for $k \rightarrow k_\varphi$ λ_k indeed develops the pole-like characteristic for a bound state and we can perform the transition to a description in terms of a linear σ -model coupled to quarks.

The next step involves the computation of the effective scalar potential from (5.20). In the present model $\Gamma^{(2)}$ contains the free fermionic propagator and the Yukawa coupling to σ_a^b as in eq. (5.18):

$$\begin{aligned} (\Gamma^{(2)}(q, q'))^a_b &= \not{q} \delta_b^a (2\pi)^4 \delta^4(q - q') \\ &+ \frac{i}{2} g(-q, q') \{ \sigma_b^a(q - q') + (\sigma^\dagger)^a_b(q' - q) + (\sigma_b^a(q - q') - (\sigma^\dagger)^a_b(q' - q)) \gamma_5 \} \end{aligned} \quad (6.8)$$

In addition $\Gamma^{(2)}$ is diagonal in colour space. We will be interested in the effective potential for the field σ_a^b . This obtains for a spatially homogeneous configuration which we will take diagonal and real:

$$\sigma_a^b(p) = \sigma \delta_a^b (2\pi)^4 \delta^4(p) \quad (6.9)$$

With (6.8) for $\Gamma^{(2)}$ and the form (5.6) of R_{kF} the equation (5.20) for the effective potential $V_k(\sigma)$ becomes

$$V_k(\sigma) = \frac{1}{2}N_f\tilde{G}^{-1}(0)\sigma^2 - 2N_fN_c \int \frac{d^4q}{(2\pi)^4} \ln \frac{P_F^{(\Lambda,k)}(q) + g^2(-q, q)\sigma^2}{P_F^{(\Lambda,k_\varphi)}(q) + g^2(-q, q)\sigma^2} \quad (6.10)$$

with

$$P_F^{(\Lambda,k)}(q^2) = \frac{q^2}{(\exp(-q^2/\Lambda^2) - \exp(-q^2/k^2))^2} \quad (6.11)$$

We observe, in particular, the k -dependence of the mass term at the origin; neglecting the q -dependence of g we have

$$\left. \frac{\partial^2 V_k}{\partial \sigma^2} \right|_{\sigma=0} = N_f\tilde{G}^{-1}(0) - \frac{N_fN_c}{8\pi^2}(k_\varphi^2 - k^2) \cdot \left(\frac{4\Lambda^4}{(\Lambda^2 + k^2)(\Lambda^2 + k_\varphi^2)} - 1 \right) \quad (6.12)$$

which becomes negative for small enough k and $\tilde{G}^{-1}(0)$. This induces spontaneous breaking of the chiral symmetry. For the model under consideration and the choice of parameters discussed before, the resulting effective potential exhibits indeed a nontrivial minimum (independent of N_f). In fig. 5 we show $V_k(\sigma)$ for $k = .63, .27, .18$ and 0 GeV. Numerically the minimum of $V_0(\sigma)$ is found to be at

$$\sigma_0 = .18 \text{ GeV} \quad (6.13)$$

Due to our convention (5.14) for $g(0, 0)$, and the $\sigma\bar{\psi}\psi$ coupling present in $\Gamma[\psi, \bar{\psi}, \sigma]$, this number can directly be interpreted as a constituent quark mass.

In order to compute the chiral condensate $\langle \bar{\psi}\psi \rangle$ from the vacuum expectation value σ_0 we first Fourier-transform the operator (5.12)

$$\begin{aligned} O_a^b(x) &= -i \int d^4y d^4z \bar{\psi}_a^i(x+y) \psi_i^b(x+z) g(y, z) \\ g(y, z) &= \int \frac{d^4q}{(2\pi)^4} \frac{d^4q'}{(2\pi)^4} \exp i(q'_\mu z^\mu - q_\mu y^\mu) g(-q, q') \end{aligned} \quad (6.14)$$

For constant $g(-q, q') = 1$ the operator O_a^b reduces to $\bar{\psi}_a^i(x) \psi_i^\beta(x)$ (up to the phase factor which appears as a consequence of our Euclidean signature convention). The smoothening due to the wave function $g(y, z)$ gives a well-defined regularized meaning¹³ to the chiral condensate $\langle \bar{\psi}\psi \rangle$ as the expectation value of O

$$\langle \bar{\psi}\psi \rangle = \frac{1}{N_f} \langle \text{Tr } O(x) \rangle \quad (6.15)$$

The exact relation of our field-theoretical definition of $\langle \bar{\psi}\psi \rangle$ to more phenomenological definitions in the context of QCD sum rules remains to be established. We emphasize that our formalism offers, in principle, the possibility to provide well-defined field-theoretical definitions for the concepts underlying QCD sum rules or

¹³ An additional smoothening will be related to the appearance of f_{k_φ} in the definition of O (4.4) in a more accurate treatment.

vacuum correlators. The relation between the *vev* of the collective field $\sigma_a^b(x)$ and the expectation value of the composite operator O_i^j is given by eq. (3.11)

$$\langle \sigma_a^b \rangle = \tilde{G}(0) \langle O_a^b \rangle. \quad (6.16)$$

Therefore the relation between σ_0 and the QCD quark condensate reads

$$\sigma_0 = \tilde{G}(0) \langle \bar{\psi}\psi \rangle. \quad (6.17)$$

With $\tilde{G}(0) = 33.2 \text{ GeV}^{-2}$ we thus find

$$\langle \bar{\psi}\psi \rangle = (175 \text{ MeV})^3. \quad (6.18)$$

It should be clear that the collective fields σ_a^b contain the massless pionic degrees of freedom, once they are expanded around the *vev* (6.9). We can also obtain the pion decay constant f_π , if we expand the effective action $\Gamma_{0,0}(0, \sigma)$ up to second order in derivatives acting on σ . Let us denote the corresponding kinetic terms in $\Gamma_{k,0}$ by

$$\frac{1}{2} \int \frac{d^4 q}{(2\pi)^4} Z_k \sigma^{\dagger a}{}_b(q) q^2 \sigma_a^b(q); \quad (6.19)$$

With this normalization we have

$$f_\pi = \sqrt{2Z_{k=0}} \cdot \sigma_0 \quad (6.20)$$

As stated below eq.(5.21), Z_0 gets contributions from both terms on the r.h.s. of eq. (5.20). In the model under consideration we actually find a dominant contribution .825 arising from the term involving \tilde{G}^{-1} ; together with a contribution .017 from the fermionic loop we thus obtain

$$f_\pi = 234 \text{ MeV} \quad (6.21)$$

which has to be compared with the experimental value $f_\pi = 93 \text{ MeV}$. We observe a substantial discrepancy, but expect that the dominant contribution arising from the term involving \tilde{G}^{-1} is particularly sensitive to the rather crude approximations employed in this paper.

7 Conclusions

We have presented in this paper a formalism to describe bound states and condensates of composite operators in terms of an effective average action. The scale dependence of this effective action is determined by an exact nonperturbative evolution equation. The properties of the vacuum and excitations around it can be inferred from the solution of the flow equation for $k \rightarrow 0$. Such a solution allows to extrapolate from the short distance physics to the long distance physics. We have devised a detailed prescription how a change of relevant variables can be performed in the course of this evolution.

The practical use of the exact flow equation is closely related to the existence of a suitable truncation scheme which permits to solve it approximately. We have demonstrated the existence of nonperturbative approximation schemes for a QCD-motivated quark model. Our computations in sect. 5 and 6 may be viewed as first steps in a systematic expansion in 1PI Green functions with a fixed number of external legs for the quark fields. Here the truncation neglects many-quark-operators but retains the full momentum dependence of the lower 1PI vertices. On the other hand the truncation for the meson part of the effective action may be developed as an expansion in momenta (or appropriate functions of momenta). This retains the full field dependence on the zero momentum modes and can therefore give a detailed picture of the meson potential and kinetic term. Although in the present paper we use very rough approximations we have demonstrated that our formalism provides not only a formal tool to describe phenomena as complex as dynamical chiral symmetry breaking, but that suitable truncations of the k -dependent effective action allow also to compute phenomenologically interesting numbers.

Here, however, no effort was made to discuss systematically the dependence of the results on various parameters entering the approximations, like Λ, k_φ or the string tension λ . We defer such an investigation to future work, where less drastic approximations to the effective action will be made. Such extensions do actually not necessarily involve an enormous amount of additional computational effort. The following parts of such a programme are certainly feasible and have partially already been performed: In the case of the four point function $\Gamma_{4,k}$ more spin, flavour and colour structures can be taken into account. After the introduction of the collective fields the k -dependence of the Yukawa-like couplings and the quark wave function normalisation do not have to be neglected.

The meson self-interactions can be included on the r.h.s. of the flow equation, using the picture with infrared cutoff for both fundamental and composite fields developed in sect. 4. The transition from the description in terms of only fundamental fields to a language with collective fields can be smoothened by an appropriate separation of the bound-state scale from the scales k_φ, Ak_φ as discussed in sect. 5. Quark mass terms can be included without any conceptual difficulties. We are also free to include the ρ -mesons as additional collective fields. Finally, we may replace the “initial conditions” for the four quark vertex at the ultraviolet cutoff scale Λ by a solution of the evolution equation for QCD, i.e. the coupled system of quarks and gluons. The practical applicability of the effective average action for nonperturbative problems in gauge theories has already been demonstrated [26]. Although this program still needs many steps it seems not impossible to us that it may finally lead to a reliable computation of chiral condensates and the meson spectrum and interactions.

Appendix A: Green functions for collective fields and composite operators

The generating functional $\Gamma_0[\varphi, \sigma]$ contains all information about Green functions for the collective field σ , and hence as well as for the composite operator \tilde{O} . The relation between the two sets of Green functions is given in this section. The generating functional for the 1PI Green functions for the collective field obtains from $\Gamma_0[\varphi, \sigma]$ as

$$\begin{aligned}\Gamma[\sigma] &= \Gamma_0[\varphi_0[\sigma], \sigma] \\ \frac{\delta\Gamma_0}{\delta\varphi}|_{\varphi_0} &= 0\end{aligned}\tag{A.1}$$

In particular, the minimum of Γ_0

$$\begin{aligned}\frac{\delta\Gamma_0}{\delta\sigma}|_{\bar{\varphi}, \bar{\sigma}} &= 0 \\ \frac{\delta\Gamma_0}{\delta\varphi}|_{\bar{\varphi}, \bar{\sigma}} &= 0\end{aligned}\tag{A.2}$$

determines the expectation values of χ and $\tilde{O}[\chi]$

$$\begin{aligned}\langle \chi^\alpha \rangle &= \bar{\varphi}^\alpha \\ \langle \tilde{O}[\chi]^i \rangle &= \bar{\sigma}^i.\end{aligned}\tag{A.3}$$

The higher correlations for the composite operator $\tilde{O}[\chi]$ are not directly given by the functional derivatives of $\Gamma[\sigma]$ with respect to σ but they are simply related to them. From (2.1) it follows immediately that the generating functional for the connected Green functions for \tilde{O} is

$$\hat{W}[J, K] = W_0[J, K] - \frac{1}{2}K^\dagger \tilde{G}K.\tag{A.4}$$

Comparing the Legendre transforms of \hat{W} and W_0 one finds that the generating functional for the 1PI Green functions for \tilde{O} is given by $\hat{\Gamma}$

$$\begin{aligned}\hat{\Gamma} &= -\hat{W} + J^\dagger \varphi + K^\dagger \hat{\sigma} \\ &= -W_0 + \frac{1}{2}K^\dagger \tilde{G}K + J^\dagger \varphi + K^\dagger \sigma - K^\dagger \tilde{G}K \\ &= \Gamma_0 - \frac{1}{2}K^\dagger \tilde{G}K\end{aligned}\tag{A.5}$$

Here we have used

$$\hat{\sigma}^i = \frac{\partial \hat{W}}{\partial K_i} = \sigma^i - (\tilde{G}K)^i\tag{A.6}$$

The generating functional for φ and $\tilde{O}[\varphi]$ is therefore obtained from Γ_0 by straightforward algebraic manipulations

$$\hat{\Gamma}[\varphi, \hat{\sigma}] = \Gamma_0[\varphi, \sigma] - \frac{1}{2} \frac{\delta \Gamma_0}{\delta \sigma^i} \tilde{G}_j^i \frac{\delta \Gamma_0}{\delta \sigma_j^*} \quad (\text{A.7})$$

$$\hat{\sigma}^i = \sigma^i - \tilde{G}_j^i \frac{\delta \Gamma_0}{\delta \sigma_j^*} \quad (\text{A.8})$$

As expected, the difference between $\hat{\Gamma}$ and Γ_0 is irrelevant for the expectation values (A.3) but it accounts for (2.14) and generalizations to higher 1PI vertices. The Green functions for O obtain from the Green functions for \tilde{O} by simple rescaling (2.11).

Appendix B: Exact evolution equation restricted to composite fields

If φ_0 depends on σ one has to correct (4.3) according to

$$\frac{d}{d\sigma} \left(\frac{\partial \Gamma}{\partial \varphi} [\varphi_0[\sigma], \sigma] \right) = \frac{\partial^2 \Gamma}{\partial \sigma \partial \varphi} [\varphi_0[\sigma], \sigma] + \frac{\partial^2 \Gamma}{\partial \varphi^2} [\varphi_0[\sigma], \sigma] \frac{d\varphi_0}{d\sigma} = 0 \quad (\text{B.1})$$

and use

$$\frac{d^2 \Gamma[\sigma]}{d\sigma^2} = \frac{\partial^2 \Gamma}{\partial \sigma^2} [\varphi_0[\sigma], \sigma] + \frac{\partial^2 \Gamma}{\partial \sigma \partial \varphi} [\varphi_0[\sigma], \sigma] \frac{d\varphi_0}{d\sigma} \quad (\text{B.2})$$

since $d/d\sigma = \partial/\partial\sigma + (d\varphi_0/d\sigma)\partial/\partial\varphi$. The evolution equation (3.7) expressing $\partial \Gamma_{\tilde{k}}[\sigma]/\partial \tilde{t}$ in terms of $\Gamma_{\tilde{k}}^{(2)}[\sigma]$ remains nevertheless exact. This can be seen most easily by using an implicit functional integral representation for $\Gamma_{\tilde{k}}[\sigma]$

$$\begin{aligned} & \exp -\Gamma_{\tilde{k}}[\sigma] \\ &= \int \mathcal{D}\chi \mathcal{D}\rho \exp - \left\{ S[\chi] + \frac{1}{2} \tilde{G}^{-1} (\rho - \tilde{G} O[\chi])^2 - \frac{\delta \Gamma_{\tilde{k}}[\sigma]}{\delta \sigma} (\rho - \sigma) + \tilde{\Delta}_{\tilde{k}} S[\rho - \sigma] \right\} \\ &= \int \mathcal{D}\rho' \exp - \left\{ S_{eff}[\rho + \rho'] - \frac{\delta \Gamma_{\tilde{k}}[\sigma]}{\delta \sigma} \rho' + \tilde{\Delta}_{\tilde{k}} S[\rho'] \right\}. \end{aligned} \quad (\text{B.3})$$

We note the complete analogy with the functional integral representation [16] of the effective average action for the fundamental fields

$$\exp -\Gamma_k[\varphi] = \int \mathcal{D}\chi' \exp - \left\{ S[\varphi + \chi'] - \frac{\delta \Gamma_k[\varphi]}{\delta \varphi} \chi' + \Delta_k S[\chi'] \right\}. \quad (\text{B.4})$$

The exact evolution equation can be inferred from this analogy by observing that the precise functional form of S_{eff} needs not to be known. Alternatively, one can derive the evolution equation starting directly from (B.4) and following the procedure of ref. [11]. One arrives at

$$\frac{\partial}{\partial t} \Gamma_{\tilde{k}}[\sigma] = \frac{1}{2} \text{Tr} \left\{ (\Gamma_{\tilde{k}}^{(2)}[\sigma] + \tilde{R}_{\tilde{k}})^{-1} \frac{\partial}{\partial t} \tilde{R}_{\tilde{k}} \right\} \quad (\text{B.5})$$

with $\Gamma_{\tilde{k}}^{(2)}$ now denoting second functional derivatives with respect to σ .

Appendix C: Effective average action with fermions

In this short section we present the generalization of our formalism for fermions. We write the fermionic infrared cutoff term as

$$\Delta_k S_F = \left(\frac{1}{2}\right) \bar{\eta}_\alpha (R_{kF})^\alpha_\beta \eta^\beta. \quad (\text{C.1})$$

Here η are Grassmann variables which are either Dirac spinors or Weyl spinors in even dimensions, or Majorana or Majorana-Weyl spinors in those dimensions where this is possible. (We use the conventions for Euclidean spinors of ref. [22].) The factor $\frac{1}{2}$ appears only if η and $\bar{\eta}$ are related, as for the case of Majorana and Majorana-Weyl spinors or a representation of Weyl spinors in the form of $2^{\frac{d}{2}}$ component spinors with identified components. For Dirac spinors or the standard $2^{\frac{d}{2}-1}$ component Weyl spinors this factor is absent. The formalism of ref. [4] applies identically for fermions except for one important minus sign. First we observe that the infrared cutoff does not mix bosons and fermions. The evolution of the effective average action $\partial\Gamma_k/\partial t$ obtains therefore separate contributions from bosons and fermions. Both for the bosonic and fermionic contribution one arrives at the identity (2.21, 2.22)

$$\begin{aligned} \frac{\partial}{\partial t} \Gamma_k &= \frac{1}{2} \frac{\partial}{\partial t} (R_{kB})^\alpha_\beta (\langle \chi_\alpha^* \chi^\beta \rangle - \langle \chi_\alpha^* \rangle \langle \chi^\beta \rangle) \\ &+ \left(\frac{1}{2}\right) \frac{\partial}{\partial t} (R_{kF})^\alpha_\beta (\langle \bar{\eta}_\alpha \eta^\beta \rangle - \langle \bar{\eta}_\alpha \rangle \langle \eta^\beta \rangle). \end{aligned} \quad (\text{C.2})$$

The minus sign for the fermionic contribution appears due to the anticommutation properties of the Grassmann variables

$$\langle \bar{\eta}_\alpha \eta^\beta \rangle - \langle \bar{\eta}_\alpha \rangle \langle \eta^\beta \rangle = -(\langle \eta^\beta \bar{\eta}_\alpha \rangle - \langle \eta^\beta \rangle \langle \bar{\eta}_\alpha \rangle) = -((\Gamma_k^{(2)} + R_k)_{FF}^{-1})^\beta_\alpha \quad (\text{C.3})$$

Here $\Gamma_k^{(2)}$ and R_k are considered as enlarged matrices acting both on bosonic and fermionic degrees of freedom, i.e.

$$\begin{aligned} (\Gamma_k^{(2)})_{FF\alpha}^\beta &= -\frac{\delta^2 \Gamma_k}{\delta \bar{\eta}_\beta \delta \eta^\alpha} = \frac{\delta^2 \Gamma_k}{\delta \eta^\alpha \delta \bar{\eta}_\beta} \\ (\Gamma_k^{(2)})_{BF\alpha}^\beta &= \frac{\delta^2 \Gamma_k}{\delta \chi_\beta^* \delta \eta^\alpha} \\ (\Gamma_k^{(2)})_{FB\alpha}^\beta &= \frac{\delta^2 \Gamma_k}{\delta \bar{\eta}_\beta \delta \chi^\alpha} \\ (\Gamma_k^{(2)})_{BB\alpha}^\beta &= \frac{\delta^2 \Gamma_k}{\delta \chi_\beta^* \delta \chi^\alpha} \end{aligned} \quad (\text{C.4})$$

(We observe that the second derivatives (C.4) are sufficient to describe $\Gamma_k^{(2)}$ only if χ, χ^* and $\eta, \bar{\eta}$ are not independent, e.g. for real scalar fields and Majorana spinors.

For Dirac spinors η and $\bar{\eta}$ lead to independent entries in the matrix of second functional derivatives and one needs, for example, $\delta^2\Gamma_k/\delta\chi_\beta^*\delta\bar{\eta}_\alpha$.) The inclusion of possible fermionic composite operators is straightforward and our final evolution equation is

$$\frac{\partial}{\partial t}\Gamma_{k,\tilde{k}} = \frac{1}{2}\text{Tr}\left\{\frac{\partial R_{kB}}{\partial t}\left(\Gamma_{k,\tilde{k}}^{(2)} + R_k\right)^{-1}\right\} - \frac{1}{2}\text{Tr}\left\{\frac{\partial R_{kF}}{\partial t}\left(\Gamma_{k,\tilde{k}}^{(2)} + R_k\right)^{-1}\right\} \quad (\text{C.5})$$

The first trace effectively runs only over bosonic degrees of freedom and the second over fermionic ones. The second functional derivative $\Gamma_k^{(2)}$ involves both fundamental and composite fields. The evolution with respect to \tilde{k} is similar (cf. (3.7)).

Figure Captions

Fig. 1: Vertex joining the collective field ρ to two fundamental fields χ as present in the action $S_k[\chi, \rho]$ of eq. (3.3).

Fig. 2: Form of the term $\sim O^\dagger[\chi]\tilde{G}O[\chi]$ in the action $S_k[\chi, \rho]$ of eq. (3.3).

Fig. 3: Diagrammatic representation of the evolution equation (5.9) for the four point function λ_k .

Fig. 4: Full line: The collective field propagator $\tilde{G}(s)$, at $s = 0$, as a function of the scale k , in units of 100 GeV^{-2} . This result has been obtained from a numerical computation of C_k as described in sect.6, and using the relation (6.7) between C_k and \tilde{G} .

Dotted line: The measure f , as defined below eq. (6.7), as a function of the scale k . The approach of f towards 1 indicates that the momentum dependence of the four point function λ_k or its Laplace transform C_k indeed factorizes as used in eq. (6.7).

The scale k_φ , where the collective field is introduced, is chosen at .63 GeV.

Fig. 5: Effective potential for the collective field σ , as given by eq.(6.10), for different values of the infrared cutoff k . The dotted lines correspond to, from inside to outside, $k = .63 \text{ GeV}$, $k = .27 \text{ GeV}$ and $k = .18 \text{ GeV}$. The full line is the final result for $k = 0$.

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